MOVEMENT OF GROUND WATER IN AN INHOMOGENEOUS FINITE AQUIFER IN THE PRESENCE OF A RESERVOIR HEAD AND IRRIGATION

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A number of problems of nonsteady gravity seepage connected with water storage and irrigation over homogeneous strata have been solved by P. Ya. Polubarinova-Kochina [1, 2], N. N. Verigin [3, 4], S. F. Aver'yanov [5], and others. However, less attention has been paid to ground-water flow in strata with variable properties, even though these are more common in nature. Of special interest [6] is the flow model in which the permeability parameters are treated as piecewise-constant over the length of the bed.

Let us divide an aquifer of finite length L into two zones of different composition: zone 1 ($0 \le x \le l_1$) with permeability parameters $k_j = k_1$ and $a_j^2 = a_1^2$, and zone 2 ($l_1 \le x \le L$) with $k_j = k_2$ and $a_j^2 = a_2^2$ (Fig. 1). The impermeable base stratum is horizontal. The depth of the steady ground-water flow under natural conditions is known from observations.

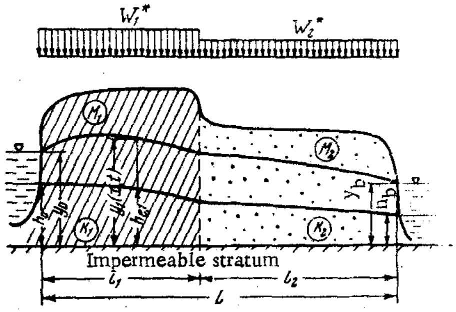

Fig. 1

Let us assume that the water table rises instantaneously from h_0 to y_0 at x = 0, and from h_b to y_b at x = L as a result of the construction of a reservoir. At the same time, because of nonuniform irrigation the ground-water flow in zone 1 is supplemented at a rate w_1^*, and in zone 2 at a rate w_2^*. We further assume that the depth of water at the outside edges of the aquifer and the rates of supplementary infiltration w_1^* and w_2^* remain constant in time.

It is required to find the depth of the ground-water flow under the given conditions.

It is known [4] that the Boussinesq equation for one-dimensional nonsteady gravity filtration over a horizontal impermeable stratum, linearized by the Bagrov-Verigin method, coincides with the equation of heat conduction and takes the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + b = \frac{\partial u}{\partial t}$$
$$\left(u = \frac{h^2}{2}, \quad a^2 = \frac{kh^\circ}{\mu}, \quad b = \frac{wh^\circ}{\mu}\right). \qquad (1)$$

Here h is the variable depth (head) of the ground water, t is time, μ is the soil saturation deficit in the aeration zone, w is the rate of intake of ground water from above, and h° is some mean flow depth. Starting from the superposition principle, we will find the solution of the problem in the form

$$y_1^2(x, t) = h_{e,1}^2 + 2u_1(x, t) \qquad (0 \le x \le l_1),$$
$$y_2^2(x, t) = h_{e,2}^2 + 2u_2(x, t) \qquad (l_1 \le x \le L). \qquad (2)$$

Here y(x, t) is the depth of the nonsteady ground-water flow in the presence of a reservoir head and irrigation, time zero t = 0 being taken as the moment at which these effects start to operate. The subscripts 1 and 2 relate to the different zones.

In accordance with (1), the functions u_1 and u_2 are given by the system of equations

$$a_1^2 \frac{\partial^2 u_1}{\partial x^2} + b_1 = \frac{\partial u_1}{\partial t} \qquad (0 \leqslant x \leqslant l_1),$$
$$\left(a_1^2 = \frac{k_1 h_1^\circ}{\mu_1}, \quad b_1 = \frac{w_1^* h_1^\circ}{\mu_1} = \frac{w_1^*}{k_1} a_1^2\right),$$
$$a_2^2 \frac{\partial^2 u_2}{\partial x^2} + b_2 = \frac{\partial u_2}{\partial t} \qquad (l_1 \leqslant x \leqslant L),$$
$$\left(a_2^2 = \frac{k_2 h_2^\circ}{\mu_2}, \quad b_2 = \frac{w_2^* h_2^\circ}{\mu_2} = \frac{w_2^*}{k_2} a_2^2\right). \qquad (3)$$

The initial condition is

$$u_1(x, 0) = 0, \qquad u_2(x, 0) = 0. \qquad (4)$$

The conditions at the outside edges of the stratum are

$$u_1(0, t) = \tfrac{1}{2}(y_0^2 - h_0^2), \qquad u_2(L, t) = \tfrac{1}{2}(y_b^2 - h_b^2). \qquad (5)$$

The conditions at the interface are

$$\left.\frac{\partial u_1}{\partial t}\right|_{x=l_1} = \left.\frac{\partial u_2}{\partial t}\right|_{x=l_1}, \qquad k_1 \left.\frac{\partial u_1}{\partial x}\right|_{x=l_1} = k_2 \left.\frac{\partial u_2}{\partial x}\right|_{x=l_1}. \qquad (6)$$

System (3) does not describe all the seepage flow, but only the additional nonsteady flow due to the reservoir head and the supplementary intake along the length of the bed. In accordance with (2), this flow is superposed on the natural steady-state ground-water flow.

We will solve the system of second-order partial differential equations (3) by an operational method [7, 8]. As a result of the Laplace transformation

$$U_j(x, p) = \int_0^\infty u_j(x, t)\, e^{-pt} dt, \qquad (7)$$

we get the representative system

$$a_1^2 \frac{d^2 U_1}{dx^2} - pU_1 + \frac{b_1}{p} = 0 \qquad (0 \leqslant x \leqslant l_1),$$
$$a_2^2 \frac{d^2 U_2}{dx^2} - pU_2 + \frac{b_2}{p} = 0 \qquad (l_1 \leqslant x \leqslant L), \qquad (8)$$

with conditions

$$U_1(0) = (y_0^2 - h_0^2)/2p, \qquad U_2(L) = (y_b^2 - h_b^2)/2p, \qquad (9)$$
$$U_1(l_1) = U_2(l_1), \qquad k_1 \left.\frac{dU_1}{dx}\right|_{x=l_1} = k_2 \left.\frac{dU_2}{dx}\right|_{x=l_1}. \qquad (10)$$

Solving system (8), we find the L-transforms of the unknown functions:

$$U_1 = \frac{1}{2p}\left[(y_0^2 - h_0^2)\frac{\operatorname{ch}[(l_1 - x)\beta_1 \sqrt{p}]}{\operatorname{ch}(\lambda_1 \sqrt{p})} + \right.$$
$$\left. + \frac{\operatorname{sh}(x\beta_1 \sqrt{p})}{\operatorname{ch}(\lambda_1 \sqrt{p})} \frac{(y_b^2 - h_b^2)\operatorname{sch}(\lambda_2 \sqrt{p}) - (y_0^2 - h_0^2)\operatorname{sch}(\lambda_1 \sqrt{p})}{\operatorname{th}(\lambda_1 \sqrt{p}) + \sigma \operatorname{th}(\lambda_2 \sqrt{p})}\right] +$$
$$+ \frac{b_1}{p^2}\left[1 - \operatorname{ch}(x\beta_1 \sqrt{p}) + \right.$$
$$\left. + \frac{\operatorname{sh}(x\beta_1 \sqrt{p})[1 - \operatorname{sch}(\lambda_1 \sqrt{p}) + \sigma \operatorname{th}(\lambda_1 \sqrt{p}) \operatorname{th}(\lambda_2 \sqrt{p})]}{\operatorname{th}(\lambda_1 \sqrt{p}) + \sigma \operatorname{th}(\lambda_2 \sqrt{p})}\right] +$$
$$+ \frac{b_2}{p^2}\left[\frac{\operatorname{sh}(x\beta_1 \sqrt{p})[1 - \operatorname{ch}(l_1\beta_2 \sqrt{p})\operatorname{sch}(L\beta_2 \sqrt{p})]}{\operatorname{sh}(\lambda_1 \sqrt{p})} + \right.$$
$$+ \frac{\operatorname{sh}(x\beta_1 \sqrt{p}) \operatorname{th}(\lambda_2 \sqrt{p})}{\operatorname{sh}(\lambda_1 \sqrt{p})} \times$$
$$\times \{\sigma[\operatorname{sch}(L\beta_2 \sqrt{p}) \operatorname{ch}(l_1\beta_2 \sqrt{p}) - 1] -$$
$$- \operatorname{th}(\lambda_1 \sqrt{p}) \operatorname{sh}(l_1\beta_2 \sqrt{p}) \operatorname{sch}(L\beta_2 \sqrt{p})\} \cdot$$
$$\left. \cdot \{\operatorname{th}(\lambda_1 \sqrt{p}) + \sigma \operatorname{th}(\lambda_2 \sqrt{p})\}^{-1}\right]. \qquad (11)$$

$$U_2 = \frac{1}{2p} \times$$

$$\times \Big[(y_0^2 - h_0^2)\sigma \frac{\operatorname{sh}[(L-x)\beta_2 \sqrt{p}]}{\operatorname{ch}(\lambda_1\sqrt{p})\operatorname{ch}(\lambda_2\sqrt{p})[\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_2\sqrt{p})]} +$$

$$+ (y_b^2 - h_b^2)\frac{\operatorname{ch}[(x-l_1)\beta_2\sqrt{p}]}{\operatorname{ch}(\lambda_2\sqrt{p})} \times$$

$$\times \frac{\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}[(x-l_1)\beta_2\sqrt{p}]}{\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_2\sqrt{p})}\Big] +$$

$$+ \frac{b_1}{p^2}\Big[\frac{\operatorname{sh}[(L-x)\beta_2\sqrt{p}][1-\operatorname{ch}(\lambda_1\sqrt{p})]}{\operatorname{sh}(\lambda_2\sqrt{p})} +$$

$$+ \{\operatorname{sh}[(L-x)\beta_2\sqrt{p}]\operatorname{sh}(\lambda_1\sqrt{p}) \times$$

$$\times [1 - \operatorname{sch}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_1\sqrt{p})\operatorname{th}(\lambda_2\sqrt{p})]\} \times$$

$$\times \{\operatorname{sh}(\lambda_2\sqrt{p})[\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_2\sqrt{p})]\}^{-1} +$$

$$+ \frac{b_2}{p^2}\Big[1 - \operatorname{ch}(x\beta_2\sqrt{p})\operatorname{sch}(L\beta_2\sqrt{p}) + \frac{\operatorname{sh}[(L-x)\beta_2\sqrt{p}]}{\operatorname{ch}(\lambda_2\sqrt{p})} \times$$

$$\times \{\sigma[\operatorname{sch}(L\beta_2\sqrt{p})\operatorname{ch}(l_1\beta_2\sqrt{p}) - 1] -$$

$$- \operatorname{th}(\lambda_1\sqrt{p})\operatorname{sh}(l_1\beta_2\sqrt{p})\operatorname{sch}(L\beta_2\sqrt{p})\} \times$$

$$\times \{\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_2\sqrt{p})\}^{-1}. \qquad (12)$$

Here

$$\sigma = \frac{k_1 a_2}{k_2 a_1}, \quad \beta_1 = \frac{1}{a_1}, \quad \beta_2 = \frac{1}{a_2}, \quad \lambda_1 = \frac{l_1}{a_1}, \quad \lambda_2 = \frac{l_2}{a_2}. \qquad (13)$$

To convert to the inverse transforms, we use the Riemann-Mellin conversion formula, according to which

$$u_j(x, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} U_j(x, p)\, e^{pt}\, dp. \qquad (14)$$

The integrals obtained by substituting in (14) the expressions for U_1 and U_2 from (11) and (12) may be evaluated by going over to a closed contour and applying the residue theorem. According to Cauchy's theorem, the determination of these integrals reduces to evaluation of the sum of the residues with respect to simple poles of the integrand corresponding to the roots of the transcendental equation

$$\operatorname{th}(\lambda_1\sqrt{p}) + \sigma\operatorname{th}(\lambda_2\sqrt{p}) = 0. \qquad (15)$$

One of the roots of (15) is $p = 0$. The remaining roots are real negative numbers $p = -\alpha^2$.

Substituting the circular for the hyperbolic tangent in (15) and taking into account its property of periodicity, we get

$$\operatorname{tg}(\alpha\lambda_1 + \pi m) + \sigma\operatorname{tg}(\alpha\lambda_2 + \pi s) = 0, \qquad (16)$$

where m and s are any integers.

Equation (16) has an infinite set of real roots $\alpha_1, \alpha_2, \alpha_3, \ldots,$ which are distributed symmetrically with respect to the coordinate origin and do not recur; to each positive root there corresponds an equal negative root.

In the general case the roots of (16) can be found graphically as the abscissas of the points of intersection of the curves

$$y = \operatorname{tg}\lambda_1 x, \qquad y = -\sigma\operatorname{tg}\lambda_2 x.$$

In a number of cases the transcendental trigonometric equation (16) reduces to an algebraic equation.

Without writing down the inverse transforms, we arrive at the final form of the equation of the curve of ground-water depression under the combined action of a bilateral reservoir head and irrigation:

in zone 1

$$y_1^2(x, t) = y_{k,1}^2 - 2a_1\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\exp(-\alpha_n^2 t)\frac{\sin(\alpha_n x/a_1)}{\cos(\alpha_n\lambda_1)} \times$$

$$\times \frac{(y_0^2 - h_0^2)\sec(\alpha_n\lambda_1) - (y_b^2 - h_b^2)\sec(\alpha_n\lambda_2)}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)} -$$

$$- 4\frac{a_1^2}{k_1}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^3}\exp(-\alpha_n^2 t)\frac{\sin(\alpha_n x/a_1)}{\cos(\alpha_n\lambda_1)} \times$$

$$\times \frac{w_1^* a_1[\sec(\alpha_n\lambda_1) - 1] - w_2^*\sigma a_2[\sec(\alpha_n\lambda_2) - 1]}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)}, \qquad (17)$$

in zone 2

$$y_2^2(x, t) = y_{k,2}^2 - 2\sigma a_1\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\exp(-\alpha_n^2 t)\frac{\sin[\alpha_n(L-x)/a_2]}{\cos(\alpha_n\lambda_2)} \times$$

$$\times \frac{(y_b^2 - h_b^2)\sec(\alpha_n\lambda_2) - (y_0^2 - h_0^2)\sec(\alpha_n\lambda_1)}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)} -$$

$$- 4\frac{a_1 a_2}{k_2}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^3}\exp(-\alpha_n^2 t)\frac{\sin[\alpha_n(L-x)/a_2]}{\cos(\alpha_n\lambda_2)} \times$$

$$\times \frac{w_2^*\sigma a_2[\sec(\alpha_n\lambda_2) - 1] - w_1^* a_1[\sec(\alpha_n\lambda_1) - 1]}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)} \qquad (\alpha_n > 0), \quad (18)$$

$$y_{k,1}^2 = (y_0^2 - h_0^2)\frac{l_1 - x + l_2 k_1/k_2}{l_1 + l_2 k_1/k_2} + (y_b^2 - h_b^2)\frac{x}{l_1 + l_2 k_1/k_2} +$$

$$+ \frac{w_1^*}{k_1}\left(\frac{l_1^2 + 2l_1 l_2 k_1/k_2}{l_1 + l_2 k_1/k_2}x - x^2\right) + \frac{w_2^*}{k_2}\frac{x l_2^2}{l_1 + l_2 k_1/k_2} + h_{e,1}^2, \qquad (19)$$

$$y_{k,2}^2 = (y_0^2 - h_0^2)\frac{(L-x)k_1/k_2}{l_1 + l_2 k_1/k_2} + (y_b^2 - h_b^2)\frac{l_1 + (x - l_1)k_1/k_2}{l_1 + l_2 k_1/k_2} +$$

$$+ \frac{w_1^*}{k_2}\frac{(L-x)l_1^2}{l_1 + l_2 k_1/k_2} + \frac{w_2^*}{k_2}\left[l_2^2\frac{l_1 + (x-l_1)k_1/k_2}{l_1 + l_2 k_1/k_2} - (x - l_1)^2\right] + h_{e,2}^2. \qquad (20)$$

The quantity y_k represents the limiting depth of ground water as $t \to \infty$ under the given conditions.

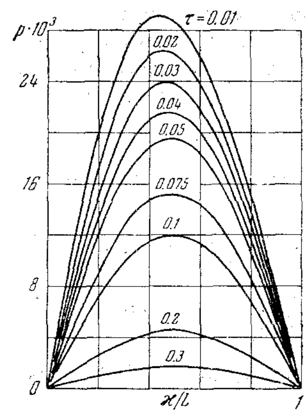

Fig. 1

Let us consider the important practical case of irrigation water infiltrating at the rate w^* over the inner part of the bed only. The length of this portion $l_{0p} = x_2 - x_1$, where x_1 and x_2 are its initial and end coordinates. Let $x_1 \geq 0$ and $x_2 = l_1$, i.e., the end of the irrigated section coincides with the interface between zones 1 and 2. The other conditions are as before.

Applying the method described, we get the following equations for the depth of flow:

in zone 1

$$y_1^2(x, t) = y_{k,1}^2 - 2a_1\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\exp(-\alpha_n^2 t)\frac{\sin(\alpha_n x/a_1)}{\cos(\alpha_n\lambda_1)} \times$$

$$\times \frac{(y_0^2 - h_0^2)\sec(\alpha_n\lambda_1) - (y_b^2 - h_b^2)\sec(\alpha_n\lambda_2)}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)} -$$

$$- 4\frac{w^*}{k_1}a_1^3\sum_{n=1}^{\infty}\frac{1}{\alpha_n^3}\exp(-\alpha_n^2 t) \times$$

$$\frac{\sin(\alpha_n x/a_1)}{\cos(\alpha_n\lambda_1)}\,\frac{\cos(\alpha_n x_1/a_1)\sec(\alpha_n\lambda_1) - 1}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)}, \qquad (21)$$

in zone 2

$$y_2^2(x, t) = y_{k,2}^2 - 2\sigma a_1\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\exp(-\alpha_n^2 t)\frac{\sin[\alpha_n(L-x)/a_2]}{\cos(\alpha_n\lambda_2)}$$

$$\times \frac{(y_b^2 - h_b^2)\sec(\alpha_n\lambda_2) - (y_0^2 - h_0^2)\sec(\alpha_n\lambda_1)}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)} -$$

$$- 4\frac{w^*}{k_1}\sigma a_1^3\sum_{n=1}^{\infty}\frac{1}{\alpha_n^3}\exp(-\alpha_n^2 t) \times$$

$$\times \frac{\sin[\alpha_n(L-x)/a_2]}{\cos(\alpha_n\lambda_2)}\,\frac{1 - \cos(\alpha_n x_1/a_1)\sec(\alpha_n\lambda_1)}{l_1\sec^2(\alpha_n\lambda_1) + k_1/k_2 l_2\sec^2(\alpha_n\lambda_2)}. \qquad (22)$$

Here

$$y_{k,1}^2=(y_0^2-h_0^2)\frac{l_1-x+l_2k_1/k_2}{l_1+l_2k_1/k_2}+(y_b^2-h_b^2)\frac{x}{l_1+l_2k_1/k_2}+$$
$$+\frac{w^*}{k_1}l_{0p}x\frac{l_{0p}+2l_2k_1/k_2}{l_1+l_2k_1/k_2}+h_{e,1}^2\quad\text{at}\quad 0\leqslant x\leqslant x_1,\quad(23)$$

$$y_{k,1}^2=(y_0^2-h_0^2)\frac{l_1-x+l_2k_1/k_2}{l_1+l_2k_1/k_2}+(y_b^2-h_b^2)\frac{x}{l_1+l_2k_1/k_2}+$$
$$+\frac{w^*}{k_1}\left[l_{0p}x\frac{l_{0p}+2l_2k_1/k_2}{l_1+l_2k_1/k_2}-(x-x_1)^2\right]+h_{e,1}^2\quad\text{at}\quad x_1\leqslant x\leqslant l_1,\quad(24)$$

$$y_{k,2}^2=(y_0^2-h_0^2)\frac{(L-x)k_1/k_2}{l_1+l_2k_1/k_2}+(y_b^2-h_b^2)\frac{l_1+(x-l_1)k_1/k_2}{l_1+l_2k_1/k_2}+$$
$$+\frac{w^*}{k_2}l_{0p}(L-x)\frac{l_{0p}+2x_1}{l_1+l_2k_1/k_2}+h_{e,2}^2\quad\text{at}\quad l_1\leqslant x\leqslant L.\quad(25)$$

It should be noted that in (21), (22), as in (17), (18), the summation is over positive roots of Eq. (16).

In the particular case of a homogeneous stratum ($k_1 = k_2$, $a_1 = a_2$, $\sigma = 1$, from (16) there follows

$$\alpha_n=\pi na/L,\quad(26)$$

where n is any positive whole number from 1 to ∞.

Substituting (26) into (21), (22), we get the equation of the curve of ground-water depression in a homogeneous finite stratum under the combined action of a bilateral reservoir head and locally intensified infiltration

$$y^2(x,t)=y_k^2-(y_b^2-h_b^2)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\pi^2n^2\tau)\sin\frac{\pi nx}{L}-$$
$$-(y_e^2-h_e^2)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\pi^2n^2\tau)\sin\frac{\pi n(L-x)}{L}-$$
$$-2\frac{w^*}{k}L^2\frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\exp(-\pi^2n^2\tau)\left[\sin\frac{\pi n(x+x_1)}{L}-\right.$$
$$\left.-\sin\frac{\pi n(x-x_2)}{L}+\sin\frac{\pi n(x-x_1)}{L}-\sin\frac{\pi n(x+x_2)}{L}\right],\quad(27)$$

where $\tau = a^2t/L^2$ is the Fourier number, dimensionless time.

Expressions for y_k for the irrigated and unirrigated parts of the bed follow from (23)–(25) if one sets $k_1 = k_2$.

The graph of the special function

$$P=\frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\exp(-\pi^2n^2\tau)\sin\frac{\pi nx}{L}$$

is shown in Fig. 2. At values $\tau > 0.3$ it is perfectly sufficient to confine oneself to the first term of the series.

The graph of the function

$$S=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\pi^2n^2\tau)\sin\frac{\pi nx}{L}$$

is given by N. N. Verigin in [9].

In conclusion, we note that the obtained solutions also extend to the case where the flow is bounded not by a reservoir but by a main irrigation ditch or a horizontal ideal drain.

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